

Haldane gap and the LSM theorem

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1 Quantization of spin

1.1 Spin coherent states

We define the element $k_z \in \text{SU}(2)/\text{U}(1) \approx \mathbb{S}^2$ by

$$k_z = \frac{1}{(1 + \bar{z}z)^{1/2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \quad (z \in \mathbb{C}). \quad (1)$$

It has a Gauss decomposition ($G_o = \bar{N}HN$)

$$k_z = e^{U_- z} e^{-U_3 \log(1 + \bar{z}z)} e^{-U_+ \bar{z}} \quad (2)$$

where $U_* = \frac{1}{2}\sigma_*$ and $\sigma_{\pm} = \sigma_1 \pm i\sigma_2$. While the LHS of Eq. (2) belongs to $SU(2)$, the individual factors in RHS are elements of $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$. Given the spin $(2j+1)$ representation $D^{(j)}$ of $SU(2)$, the decomposition (2) enables one to define the operator

$$W(\bar{z}, z) := D^{(j)}(k_z) = e^{D^{(j)}(U_-)z} e^{-D^{(j)}(U_3) \log(1+\bar{z}z)} e^{-D^{(j)}(U_+) \bar{z}}, \quad (3)$$

which generalizes the Weyl operator for the Heisenberg-Weyl group. We also use the usual notation $D^{(j)}(U_i) = \hat{J}_i$.

Let $|0\rangle$ be the highest weight state of $D^{(j)}$, and define the coherent states $|z\rangle$ labeled by $z \in \mathbb{C}$ as

$$W(\bar{z}, z)|0\rangle = \frac{1}{(1+\bar{z}z)^j} e^{\hat{J}_- z} |0\rangle = \frac{1}{(1+\bar{z}z)^j} |z\rangle. \quad (4)$$

We denote the same state as

$$|e\rangle := W(\bar{z}, z)|0\rangle = \frac{1}{(1+\bar{z}z)^j} |z\rangle, \quad z = \frac{e^1 + ie^2}{1+e^3}, \quad (e^1)^2 + (e^2)^2 + (e^3)^2 = 1, \quad (5)$$

where $e \in \mathbb{S}^2$ is the unit vector that corresponds to z under the stereographic projection. In terms of the $\hat{J}_z$ eigenstates,

$$|z\rangle = \sum_{m=-j}^j c_m(z) |m\rangle \quad c_m(z) = \left[\frac{(2j)!}{(j-m)!(j+m)!} \right]^{1/2} z^{j-m} \quad (6)$$

so that, using the notation $\langle \bar{z} | := |z\rangle^\dagger$, the overlap between two coherent states is given by

$$\langle \bar{z} | z' \rangle = (1 + \bar{z}z')^{2j}, \quad \langle e | e' \rangle = \left(\frac{1 + e \cdot e'}{2} \right)^j \left(\frac{1 + \bar{z}z'}{1 + z\bar{z}'} \right)^j. \quad (7)$$

Furthermore, the coherent states satisfy the completeness relation

$$\int d(\bar{z}, z) \frac{1}{(1+|z|^2)^{2j}} |z\rangle \langle \bar{z}| = 1, \quad d(\bar{z}, z) = \frac{d\bar{z} \wedge dz}{2i} \frac{2j+1}{\pi(1+|z|^2)^2}, \quad (8a)$$

$$\int dx |e\rangle \langle e| = 1, \quad de = \frac{2j+1}{4\pi} \sin \theta d\theta d\varphi. \quad (8b)$$

1.2 Spinor path integral

1.2.1 Coherent state path integral

$$\begin{aligned} & \langle e'' | T \exp \left[-i \int_{t'}^{t''} dt \hat{H}(t) \right] | e' \rangle \\ &= \lim_{N \rightarrow \infty} \int de_N \cdots \int de_1 \prod_{n=0}^N \langle e_{n+1} | e_n \rangle \exp [-i \Delta t_n H(e_{n+1}, e_n; t_n)] \end{aligned} \quad (9)$$

where the classical Hamiltonian is given by

$$H(e_{n+1}, e_n; t_n) := \frac{\langle e_{n+1} | \hat{H}(t_n) | e_n \rangle}{\langle e_{n+1} | e_n \rangle}. \quad (10)$$

With $\Delta e_n := e_{n+1} - e_n$, we have

$$\langle e_{n+1} | e_n \rangle = 1 + \langle \Delta e_n | e_n \rangle \approx \exp \langle \Delta e_n | e_n \rangle. \quad (11)$$

This leads to

$$\begin{aligned} & \langle e'' | T \exp \left[-i \int_{t'}^{t''} dt \hat{H}(t) \right] | e' \rangle \\ &= \lim_{N \rightarrow \infty} \int de_N \cdots \int de_1 \exp i \left[\sum_{n=0}^N -i \langle \Delta e_n | e_n \rangle - \Delta t_n H(e_{n+1}, e_n; t_n) \right] \end{aligned} \quad (12)$$

In the continuum notation,

$$\langle e'' | T e^{-i \int_{t'}^{t''} dt \hat{H}(t)} | e' \rangle = \int_{e'}^{e''} \mathcal{D}e e^{i \int_{t'}^{t''} dt \langle e | i \partial_t - \hat{H} | e \rangle}. \quad (13)$$

1.2.2 The Wess–Zumino term

To find the kinetic term of the classical Lagrangian $\mathcal{L} = \langle e | i \partial_t - \hat{H} | e \rangle$, we need to evaluate $\langle \Delta e_n | e_n \rangle$. To this end, note that (7) implies

$$\begin{aligned} \langle \Delta e | e \rangle &= \left(\frac{1 + e \cdot e'}{2} \right)^j \left(\frac{1 + \bar{z} z'}{1 + z \bar{z}'} \right)^j - 1 \\ &= \left(1 + \frac{\Delta e \cdot e}{2} \right)^j \left(1 + \frac{\bar{z} \Delta z - z \Delta \bar{z}}{1 + \bar{z} z} + \cdots \right)^j - 1 \\ &\approx \cancel{j \frac{1}{2} \Delta e \cdot e} + j \frac{\bar{z} \Delta z - z \Delta \bar{z}}{1 + \bar{z} z} \\ &= ji(1 - \cos \theta) \Delta \varphi, \end{aligned} \quad (14a)$$

$$-i \langle de | e \rangle = j(1 - \cos \theta) d\varphi = j \int_0^\theta \sin \theta' d\theta' d\varphi. \quad (14b)$$

The integral in (14b) represents the area of the infinitesimal triangle with vertices $e, e + de$, and $(0, 0, 1)$. Equivalently, we can extend the trajectory

$$e(t) = (\sin \theta(t) \cos \varphi(t), \sin \theta(t) \sin \varphi(t), \cos \theta(t)) \quad (15)$$

to a two-parameter function

$$\tilde{e}(s, t) = (\sin s \theta(t) \cos \varphi(t), \sin s \theta(t) \sin \varphi(t), \cos s \theta(t)) \quad (16)$$

so that the kinetic term for spin is given by the *Wess–Zumino term*,

$$\lim_{N \rightarrow \infty} -i \sum_{n=0}^{\infty} \langle \Delta e_n | e_n \rangle = j \int_{t'}^{t''} dt \int_0^1 ds e \cdot (\partial_t e \times \partial_s e) = j \cdot S_{\text{WZ}}[e], \quad (17a)$$

$$S_{\text{WZ}}[e] = \int_{\mathbb{S}^2} \tilde{e} \cdot (d\tilde{e} \times d\tilde{e}). \quad (17b)$$

In later sections, we will abbreviate $\tilde{e}$ simply by e .

1.2.3 Classical equation of motion

Proposition 1.1. *The variation of the Wess–Zumino term*

$$S_{\text{WZ}}[e] = \int dt \int_0^1 ds e \cdot (\partial_t e \times \partial_s e) \quad (18)$$

is given by

$$\frac{\delta S_{\text{WZ}}}{\delta e} = e \times \partial_t e. \quad (19)$$

Proof.

$$\begin{aligned} & \delta[e \cdot (\partial_t e \times \partial_s e)] \\ &= \delta e \cdot (\partial_t e \times \partial_s e) + \partial_s \delta e \cdot (e \times \partial_t e) + \partial_t \delta e \cdot (\partial_s e \times e) \\ &= \delta e \cdot (\partial_t e \times \partial_s e) + \partial_s [\delta e \cdot (e \times \partial_t e)] - \delta e \cdot \partial_s (e \times \partial_t e) - \delta e \cdot \partial_t (\partial_s e \times e) + \partial_t (\cdots) \\ &= \partial_s [\delta e \cdot (e \times \partial_t e)] + \delta e \cdot [(\partial_t e \times \partial_s e) - \partial_s (e \times \partial_t e) - \partial_t (\partial_s e \times e)] + \partial_t (\cdots) \\ &= \partial_s [\delta e \cdot (e \times \partial_t e)] + 3\delta e \cdot (\partial_t e \times \partial_s e) + \partial_t (\cdots), \end{aligned} \quad (20)$$

$$\begin{aligned} & \int_0^1 ds \delta[e \cdot (\partial_t e \times \partial_s e)] \\ &= \delta e \cdot (e \times \partial_t e) \Big|_{s=0}^{s=1} + 3 \int_0^1 ds \delta e \cdot (\partial_t e \times \partial_s e) + \partial_t (\cdots) \end{aligned} \quad (21)$$

Now, upon integration $\int dt$, the total derivative $\partial_t(\cdots)$ drops out. Furthermore, the three tangent vectors $\{\delta e, \partial_t e, \partial_s e\}$ of $\mathbb{S}^2$ are linearly dependent, hence the second term vanishes. Finally, $\delta e = 0$ on $s = 0$. This yields the desired result. $\square$

Proposition 1.2. *Let $H(e)$ be a classical Hamiltonian for a single classical spin variable e , defined by $jH(e) = \langle e | \hat{H} | e \rangle$. Then the classical equation of motion is*

$$\dot{e} = \nabla_e H \times e. \quad (22)$$

Proof. We are looking for a stationary point for the action

$$S[e] = jS_{\text{WZ}}[e] - j \int dt H(e). \quad (23)$$

By Proposition 1.1, the variation of the action is

$$\frac{1}{j} \frac{\delta S[e]}{\delta e} = e \times \dot{e} - \nabla_e H(e), \quad (24)$$

so we get an equation of motion $e \times \dot{e} = \nabla_e H$. Then using $e \cdot \dot{e} = 0$,

$$\dot{e} = (e \times \dot{e}) \times e = \nabla_e H \times e. \quad (25)$$

$\square$

Example 1.3. With spin angular momentum operators $\hat{J}_a$, consider the Hamiltonian

$$\hat{H} = B \cdot \hat{J} = B^a \hat{J}_a \quad (26)$$

where B^a is interpreted as a magnetic field. Note that (26) is the most general form of Hamiltonian if $j = 1/2$. Since the classical Hamiltonian is

$$H(e) = j^{-1} \langle e | \hat{H} | e \rangle = j^{-1} B^a \langle e | \hat{J}_a | e \rangle = B^a e_a, \quad (27)$$

the classical motion is the familiar precession.

$$\dot{e} = B \times e \quad (28)$$

2 Spin chain models

In this section, we analyze the Heisenberg model in one dimension (1D) semiclassically, using the spinor path integral established in Sec. 1.2. The quantum and classical Hamiltonians are given by

$$\hat{H} = \mp \frac{J}{2j} \sum_{\langle xx' \rangle} \hat{S}(x) \cdot \hat{S}(x') - \sum_x B^a \hat{S}_a(x) \quad (J > 0), \quad (29a)$$

$$H(e) = \mp \frac{J}{2j} \sum_{\langle xx' \rangle} S(x) \cdot S(x') - \sum_x B^a S_a(x) \quad (J > 0). \quad (29b)$$

where $\langle xx' \rangle$ runs over the adjacent pair of lattice sites. The minus (plus) sign before the coupling J indicates that the model is ferromagnetic (antiferromagnetic). Together with the Wess–Zumino term, the action is given by

$$\frac{1}{j} S[e] = \int dt \left\{ \int_0^1 ds \sum_x e(x) \cdot (\partial_t e(x) \times \partial_s e(x)) \pm \frac{J}{2} \sum_{\langle xx' \rangle} e(x) \cdot e(x') - \sum_x e(x) \cdot B \right\} \quad (30)$$

The continuum approximation of the coupling term can be obtained by using the Taylor expansion

$$e(x \pm a\hat{i}) = e(x) \pm a \nabla_i e(x) + \frac{1}{2} a^2 \nabla_i^2 e(x) + O(a^3). \quad (31)$$

Integrating by parts and ignoring constant terms, the coupling term can be replaced by

$$\sum_{\langle xx' \rangle} e(x) \cdot e(x') = \sum_x \sum_i \sum_{x'=x \pm a\hat{i}} e(x) \cdot e(x') \mapsto -a^2 \sum_x \nabla e(x) \cdot \nabla e(x). \quad (32)$$

The resulting continuum action is the integral of the Lagrangian density, $j^{-1} S[e] = \int dt dx \mathcal{L}[e]$, with

$$\mathcal{L}[e] = \int_0^1 ds e \cdot (\partial_t e \times \partial_s e) \mp \frac{J}{2} \nabla e \cdot \nabla e - e \cdot B \quad (33)$$

2.1 Ferromagnetic Heisenberg model

For a ferromagnet, we take the upper sign in front of the J -term in (33). Applying Proposition 1.2 to the Hamiltonian density (which can be read off from (33)),

$$\mathcal{H}[e] = \frac{J}{2} \nabla e \cdot \nabla e + e \cdot B, \quad (34)$$

we obtain an equation of motion

$$\dot{e} = J e \times \Delta e + B \times e. \quad (35)$$

Equation (35) is known as the *Landau–Lifschitz equation*.

2.2 Antiferromagnetic Heisenberg model

For an antiferromagnet, we take the lower sign in (33). Due to the positive coupling, one expects the ground state to have a Néel order where the spins are staggered. Anticipating this, we define a new variable $n(x)$ by

$$e(x) = (-1)^x n(x), \quad (36)$$

where $n(x) \in \mathbb{S}^2$ is supposed to be a smooth function of x in the continuum limit. The coupling term changes sign upon rewriting the action in terms of the new variable $n(x)$:

$$\sum_{\langle xx' \rangle} e(x) \cdot e(x') = - \sum_{\langle xx' \rangle} n(x) \cdot n(x') \quad (37)$$

This flips the sign of the coupling term back to negative. Therefore, it is the Wess–Zumino term that distinguishes the behavior of antiferromagnet from ferromagnet:

$$\sum_x e(x) \cdot (\partial_t e(x) \times \partial_s e(x)) = \sum_x (-1)^x n(x) \cdot (\partial_t n(x) \times \partial_s n(x)). \quad (38)$$

A peculiar feature of (38) is that, upon decomposing the field $n(x)$ into a slowly varying staggered field and a smooth fluctuation and then integrating out the latter, a topological term is generated; see [Gre13]. The (Euclidean) effective action for the slowly varying field is

$$S_{\text{eff}}[n] = \frac{1}{2g^2} \int d^2x \partial_\mu n \cdot \partial^\mu n + i\theta Q[n], \quad (39a)$$

$$Q[n] = \frac{1}{8\pi} \int d^2x \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n), \quad (39b)$$

$$\theta = 2\pi j. \quad (39c)$$

(39c) indicates that the topological term has no effects for the integer spin case $j \in \mathbb{Z}$. However, as will be seen in Sec. 3.3, the topological term drastically changes the long wavelength behavior of the system for half-odd-integer j .

3 Renormalization group analysis

3.1 SO(3) nonlinear sigma model

In the Euclidean signature,

$$Z = \int \mathcal{D}\mathbf{n} \delta(\mathbf{n}(x)^2 - 1) \exp \left[-\frac{1}{2g^2} \int d^d x \nabla \mathbf{n} \cdot \nabla \mathbf{n} \right] \quad (40)$$

Decompose the field $n^a(x)$ into a slowly varying background $\tilde{n}^a(x)$ and a transversal fluctuation $\chi_\alpha(x)$:

$$n^a(x) = \tilde{n}^a(x)(1 - \chi(x)^2)^{1/2} + \sum_{\alpha=1}^{N-1} \chi_\alpha(x) e_\alpha^a(x) \quad (\chi(x)^2 \ll 1) \quad (41)$$

We will expand the action in $\chi_\alpha(x)$ and only keep those terms up to quadratic order. First,

$$\begin{aligned} \nabla n^a &= \nabla \tilde{n}^a (1 - \chi^2)^{1/2} + \tilde{n}^a \nabla (1 - \chi^2)^{1/2} + \nabla (\chi_\alpha e_\alpha^a) \\ &= \nabla \tilde{n}^a (1 - \frac{1}{2} \chi^2) - \tilde{n}^a \chi_\alpha \nabla \chi_\alpha + (\nabla \chi_\alpha) e_\alpha^a + \chi_\alpha \nabla e_\alpha^a + O(\chi^2) \\ (\nabla n)^2 &= (\nabla \tilde{n})^2 (1 - \chi^2) + 2 \nabla \tilde{n}^a \nabla (\chi_\alpha e_\alpha^a) + (\nabla \chi_\alpha)^2 \\ &\quad + \chi_\alpha \chi_\beta \nabla e_\alpha^a \nabla e_\beta^a + 2 (\nabla \chi_\alpha) \chi_\beta e_\alpha^a \nabla e_\beta^a + O(\chi^3) \end{aligned} \quad (42)$$

where we used $(\nabla \tilde{n}^a) \tilde{n}_a = 0$. The partition function is now

$$Z = \int \mathcal{D}\tilde{n} \delta(\tilde{n}^2 - 1) e^{-\frac{1}{2g^2} \int d^d x (\nabla \tilde{n})^2} \int \mathcal{D}\chi e^{-\frac{1}{2g^2} \int d^d x (\nabla \chi)^2} e^{-F_I[\tilde{n}, \chi]}, \quad (43a)$$

$$F_I = \frac{1}{2g^2} \int d^d x \left[-\chi^2 (\nabla \tilde{n})^2 + 2 \nabla \tilde{n}^a \nabla (\chi_\alpha e_\alpha^a) + \chi_\alpha \chi_\beta \nabla e_\alpha^a \nabla e_\beta^a + 2 (\nabla \chi_\alpha) \chi_\beta e_\alpha^a \nabla e_\beta^a \right]. \quad (43b)$$

Now, the contribution of the fluctuating field χ ,

$$\langle e^{-F_I} \rangle_\chi = \int \mathcal{D}\chi e^{-\frac{1}{2g^2} \int d^d x (\nabla \chi)^2} e^{-F_I[\tilde{n}, \chi]} = 1 - \langle F_I \rangle_\chi + \frac{1}{2} \langle F_I^2 \rangle_\chi - \dots, \quad (44)$$

may contains kinetic terms $\sim (\nabla \tilde{n})^2$ and hence g^2 undergoes RG running. At leading order, we only need to keep $\langle F_I \rangle_\chi$:

$$\langle F_I \rangle_\chi = \frac{1}{2g^2} \int d^d x \left[-\delta_{\alpha\beta} (\nabla \tilde{n})^2 + \nabla e_\alpha^a \nabla e_\beta^a \langle \chi_\alpha(x) \chi_\beta(x) \rangle_\chi \right] \quad (45)$$

Using the relations

$$\langle \chi_\alpha(x) \chi_\beta(x) \rangle_\chi = g^2 \delta_{\alpha\beta} \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} =: g^2 \delta_{\alpha\beta} I_d, \quad (46a)$$

$$\begin{aligned} \nabla e_\alpha^a \nabla e_\alpha^a &= \nabla e_\alpha^a \nabla e_\alpha^b (\tilde{n}^a \tilde{n}^b + e_\beta^a e_\beta^b) \\ &= e_\alpha^a e_\alpha^b (\nabla \tilde{n}^a) (\nabla \tilde{n}^b) + (e_\beta^a \nabla e_\alpha^a) (e_\beta^b \nabla e_\alpha^b) \\ &= (\nabla \tilde{n}^a) (\nabla \tilde{n}^a) + (e_\beta^a \nabla e_\alpha^a)^2, \end{aligned} \quad (46b)$$

one has

$$\begin{aligned}\langle F_I \rangle_\chi &= (-\delta_{\alpha\beta}\delta^{\alpha\beta} + 1)I_d \int d^d x \frac{1}{2}(\nabla \tilde{n})^2 \\ &= (2 - N)I_d \int d^d x \frac{1}{2}(\nabla \tilde{n})^2\end{aligned}\tag{47}$$

and after rescaling of all lengths by $1/\zeta$ (so that the momentum cutoff is set back to Λ),

$$\frac{1}{g^2(\Lambda/\zeta)} = \zeta^{d-2} \left[\frac{1}{g^2(\Lambda)} + (2 - N)I_d \right]\tag{48}$$

From (48) we can compute the beta function of g . First, the integral I_d is evaluated to be

$$I_d = \frac{\Omega_{d-1}}{(2\pi)^d} \Lambda^{d-2} \left[\frac{\zeta^{2-d} - 1}{2 - d} \right], \quad \Omega_{d-1} = \text{Vol}(\mathbb{S}^{d-1}).\tag{49}$$

In particular, in $d = 2$ dimensions,

$$\frac{1}{g^2(\Lambda/\zeta)} = \frac{1}{g^2(\Lambda)} + (2 - N)\frac{1}{2\pi} \log \zeta,\tag{50}$$

$$\beta(g) = \frac{dg(\Lambda/\zeta)}{d \log \zeta} = (N - 2)\frac{g^3}{4\pi}.\tag{51}$$

Therefore, the beta function is positive for $N \geq 3$ and the coupling g^2 gets larger and larger as we flow to the infrared.

3.2 Large N limit

Consider the $\text{SO}(N)$ model which extends the case $N = 3$.

$$Z = \int \mathcal{D}\mathbf{n} \mathcal{D}\sigma \exp \left[-\frac{1}{2g^2} \int d^d x \{ \nabla n^a \nabla n_a + i\sigma(n^a n_a - 1) \} \right]\tag{52}$$

Integrate out the $n^a(x)$ fields to get an effective action for $\sigma(x)$:

$$\begin{aligned}Z &= \int \mathcal{D}\mathbf{n} \mathcal{D}\sigma \exp \left[-\int d^d x \frac{1}{2g^2} n^a (-\Delta + i\sigma) n_a \right] \exp \left[\frac{i}{2g^2} \int d^d x \sigma \right] \\ &\propto \int \mathcal{D}\sigma [\det(-\Delta + i\sigma)]^{-N/2} \exp \left[\frac{i}{2g^2} \int d^d x \sigma \right] \\ &= \int \mathcal{D}\sigma \exp \left[-\frac{N}{2} \text{Tr} \log(-\Delta + i\sigma) + \frac{i}{2g^2} \int d^d x \sigma \right]\end{aligned}\tag{53a}$$

$$S_{\text{eff}}[\sigma] = \frac{N}{2} \text{Tr} \log(-\Delta + i\sigma) - \frac{i}{2g^2} \int d^d x \sigma\tag{53b}$$

We will investigate the limit $N \rightarrow \infty$. In this limit, the majority of contributions of the path integral is concentrated near the saddle point in the function space of $\sigma(x)$. Denoting $G_\sigma(x, x') := (-\Delta + i\sigma(x))^{-1}$,

$$\frac{\delta}{\delta \epsilon} S_{\text{eff}}[\sigma + \epsilon \delta \sigma] = \frac{i}{2} N \text{Tr}[G_\sigma \delta \sigma] - \frac{i}{2g^2} \int d^d x \delta \sigma(x)\tag{54a}$$

$$\frac{\delta^2}{\delta \epsilon^2} S_{\text{eff}}[\sigma + \epsilon \delta \sigma] = \frac{N}{4} \text{Tr}[G_\sigma \delta \sigma G_\sigma \delta \sigma]\tag{54b}$$

For $\sigma = \sigma_0(x)$ to be a saddle, the Hessian $\text{Tr}[G_\sigma \delta\sigma G_\sigma \delta\sigma]$ must be strictly positive definite. The first implication of this fact is that σ_0 must be translation invariant, i.e. constant in space: Otherwise, translation invariance of the effective action S_{eff} would imply

$$S_{\text{eff}}[\sigma(\cdot + a)] = S_{\text{eff}}[\sigma(\cdot)] \quad \Rightarrow \quad \text{Hessian has a zero eigenvalue.} \quad (55)$$

The way out is to assume that the null vector $\delta_\epsilon \sigma = \sigma_0(\cdot + \epsilon) - \sigma_0(\cdot)$ is itself zero, i.e. σ_0 is invariant under translation. Thus, let us assume the constancy of $\sigma_0(x) \equiv -i\mu^2$. The Hessian can then be evaluated using the momentum bases.

$$\begin{aligned} \text{Hess} &\propto \int \frac{dp}{(2\pi)^d} \frac{dk}{(2\pi)^d} \frac{dk'}{(2\pi)^d} \frac{dk''}{(2\pi)^d} \langle p|G_\sigma|k\rangle \langle k|\delta\sigma|k'\rangle \langle k'|G_\sigma|k''\rangle \langle k''|\delta\sigma|p\rangle \\ &= \int \frac{dp}{(2\pi)^d} \frac{dp'}{(2\pi)^d} \frac{1}{p^2 + \mu^2} \frac{1}{(p')^2 + \mu^2} \widehat{\delta\sigma}(p - p') \widehat{\delta\sigma}(p' - p) \\ &= \int \frac{dk}{(2\pi)^d} \underbrace{\frac{dp}{(2\pi)^d} \frac{1}{p^2 + \mu^2} \frac{1}{(p+k)^2 + \mu^2}}_{=:\Pi(k)} \widehat{\delta\sigma}(k) \widehat{\delta\sigma}(-k) \\ &= \int \frac{dk}{(2\pi)^d} \Pi(k) |\widehat{\delta\sigma}(k)|^2 > 0 \end{aligned} \quad (56)$$

where $\mu \in \mathbb{R}$ is implicit (see below) and we used

$$\langle p|G_\sigma|k\rangle = \frac{1}{p^2 + \mu^2} (2\pi)^d \delta^{(d)}(p - k), \quad (57a)$$

$$\langle p|\delta\sigma|k\rangle = \int d^d x \delta\sigma(x) e^{-i(p-k)x} = \widehat{\delta\sigma}(p - k), \quad (57b)$$

$$\widehat{\delta\sigma}(-k) = \widehat{\delta\sigma}(k) \quad \because \delta\sigma(x) \in \mathbb{R}. \quad (57c)$$

This shows that a constant σ_0 qualifies as a saddle point. Now, it remains to compute the value of μ^2 by setting (54a) to zero:

$$\text{Tr } G_{\sigma_0} = \int^\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \mu^2} = \frac{1}{g^2 N}. \quad (58)$$

In $d = 2$ dimensions, (58) leads to

$$\frac{1}{4\pi} \log \left(\frac{\Lambda^2 + \mu^2}{\mu^2} \right) = \frac{1}{g^2 N} \quad \Rightarrow \quad \mu^2 = \frac{\Lambda^2}{e^{4\pi/g^2 N} - 1}. \quad (59)$$

This justifies a posteriori the assumption that μ is real. Also, the result (59) indicates that the field $\mathbf{n}(x)$ remains massive ($\mu^2 > 0$) in the strongly coupled deep IR limit.

Remark 3.1. *Curiously, the saddle $\sigma_0(x) = -i\mu^2$ turns out to be imaginary, but there is no problem in (52) when we used*

$$\delta(\mathbf{n}^2 - 1) \propto \int \mathcal{D}\sigma \exp \left[-\frac{i}{2g^2} \sigma(\mathbf{n}^2 - 1) \right], \quad (60)$$

because the only field which needs reality is $\delta\sigma$: Plugging $\sigma = -i\mu^2 + \delta\sigma$,

$$\delta(\mathbf{n}^2 - 1) \propto \int \mathcal{D}\delta\sigma \exp \left[-\frac{\mu^2}{2g^2} (\mathbf{n}^2 - 1) \right] \exp \left[-\frac{i}{2g^2} \delta\sigma(\mathbf{n}^2 - 1) \right]. \quad (61)$$

3.3 The role of topological term

We observed in Sec. 2.2 that the antiferromagnetic (AF) Heisenberg chain with half integer spin acquires a topological term

$$Q[\mathbf{n}] = \frac{1}{4\pi} \int d^2x \epsilon_{abc} n^a dn^b \wedge dn^c \quad (62)$$

in the continuum limit.

We lift the field $\mathbf{n}(x) \in \mathbb{S}^2$ to a $\text{SU}(2)$ -valued field $g(x)$, using the Hopf fibration $\text{SU}(2) \rightarrow \mathbb{S}^2$.

$$n^a = \frac{1}{2} \text{tr}(\sigma^a g \sigma^3 g^{-1}), \quad g = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in \text{SU}(2). \quad (63)$$

Then the action functional can be rewritten in terms of the $\text{SU}(2)$ field:

$$S[\mathbf{n}] = \frac{1}{2\lambda} \int d^2x \partial_\mu n^a \partial^\mu n^a + i\theta Q[\mathbf{n}], \quad \theta = 2\pi j \quad (64)$$

$$\rightsquigarrow S[g] = \frac{1}{2\lambda} \int d^2x \text{tr}(\partial_\mu g^{-1} \partial^\mu g) + ik \Gamma_{\text{WZ}}[g], \quad k = 2j. \quad (65)$$

where the kinetic term along the $\text{U}(1)$ fiber direction is ignored since it can be absorbed in the current-current coupling $\mathbf{J}_L \cdot \mathbf{J}_R$, to be discussed below. In other words, we are embedding the observable algebra of the $\mathbb{S}^2$ model into that of the $\text{SU}(2)$ sigma model (65). In particular, their renormalization group flow coincides, as long as one focuses on those observables that respect the full symmetry of the embedded system—the antiferromagnetic Heisenberg chain.

For a particular value $\lambda = \lambda_*$ of the coupling strength, the model (65) coincides with the $\text{SU}(2)_{k=2j}$ Wess–Zumino–Witten (WZW) conformal field theory. The deviation from the WZW fixed point is proportional to the current-current coupling:

$$S[g] = S_{\text{SU}(2)_{2j} \text{ WZW}} + \lambda' \int d^2x \mathbf{J}_L \cdot \mathbf{J}_R, \quad \lambda' \propto \lambda - \lambda_*. \quad (66)$$

It turns out that $\mathbf{J}_L \cdot \mathbf{J}_R$ is the only $\text{SU}(2)$ - and translation-invariant scalar, hence the only deformation that potentially drives the AF Heisenberg chain out of the conformal fixed point. However, it is marginally irrelevant.

Moreover, for $k > 1$ odd, the $\text{SU}(2)_k$ WZW model is not a stable fixed point—there are operators that destabilize it. The ultimate destination of the RG flow turns out to be the $\text{SU}(2)_1$ WZW model, which is a stable fixed point. Thus, without fine tuning, the uniform AF Heisenberg chain (half-odd-integer j) flows to the $\text{SU}(2)_1$ WZW CFT in the IR (central charge $c = 1$), with only logarithmic corrections from the marginally irrelevant $\mathbf{J}_L \cdot \mathbf{J}_R$. This finally reveals that the AF Heisenberg chain with half integer spin is gapless, in stark contrast with the integer spin case.

4 The Lieb-Schultz-Mattis theorem

We follow the functional-analytic approach of [Tas22]. The system of our interest is the quantum spin chain, by which we mean a tensor product Hilbert space and corresponding observable

algebra

$$\mathcal{H} = \bigotimes_{n \in \mathbb{Z}} \mathbb{C}^{2j+1}, \quad \mathfrak{A} = \bigotimes_{n \in \mathbb{Z}} \text{Mat}_{2j+1, 2j+1}(\mathbb{C}) \quad (67)$$

For each decomposable element $\hat{A} = \bigotimes_n A_n$ we define the *support* of $\hat{A}$ as

$$\text{Supp } \hat{A} = \{n \in \mathbb{Z} \mid A_n \neq I\}. \quad (68)$$

For general elements, we define its support by the set of sites x on which the operator acts nontrivially.

Definition 4.1. A local operator is an observable $\hat{A} \in \mathfrak{A}$ with finite support.

Definition 4.2 (state). A state is a linear functional $\rho : \mathfrak{A}_{\text{loc}} \rightarrow \mathbb{C}$ such that $\rho(\hat{1}) = 1$ and $\rho(\hat{A}^\dagger \hat{A}) \geq 0$ for any $\hat{A} \in \mathfrak{A}_{\text{loc}}$.

Definition 4.3 (ground state). A ground state of $\hat{H}$ is a state ω such that

$$\omega(\hat{V}^\dagger [\hat{H}, \hat{V}]) \geq 0 \quad (69)$$

for any $\hat{V} \in \mathfrak{A}_{\text{loc}}$. It follows from the definition that $|\rho(\hat{A})| \leq \|\hat{A}\|$ and $\rho(\hat{A}^\dagger) = \rho(\hat{A})^*$.

Definition 4.4 (locally unique gapped ground state). A ground state ω of $\hat{H}$ is called a locally unique gapped ground state if there exists a constant $\gamma > 0$ such that

$$\omega(\hat{V}^\dagger [\hat{H}, \hat{V}]) \geq \gamma \omega(\hat{V}^\dagger \hat{V}) \quad (70)$$

for any $\hat{V} \in \mathfrak{A}_{\text{loc}}$ with $\omega(\hat{V}) = 0$. We call the largest such γ the energy gap ΔE .

Theorem 4.5. A locally unique gapped ground state is a pure split state.

4.1 The original LSM theorem

We consider the Hamiltonians that are sum of the local terms $\hat{h}_j$, where each term satisfies

$$(\hat{U}_\theta^I)^\dagger \hat{h}_j \hat{U}_\theta^I = \hat{h}_j, \quad \hat{U}_\theta^I = \exp \left[-i \sum_{j \in I} \theta \hat{S}_j^{(z)} \right], \quad (71)$$

where I is a finite interval that contains the support of $\hat{h}_j$.

Example 4.6 (XXZ model). The XXZ model under uniform magnetic field is given by the Hamiltonian

$$\hat{H}_{\text{XXZ}} = \sum_{j \in \mathbb{Z}} J(\hat{S}_j^{(x)} \hat{S}_{j+1}^{(x)} + \hat{S}_j^{(y)} \hat{S}_{j+1}^{(y)}) + J' \hat{S}_j^{(z)} \hat{S}_{j+1}^{(z)} - H \hat{S}_j^{(z)}. \quad (72)$$

Given such a Hamiltonian and a pair $(x, l) \in \mathbb{R} \times \mathbb{R}_{>}$, we introduce the twist parameter θ_j for $j \in \mathbb{Z}$,

$$\theta_j = \begin{cases} 0 & (j < x), \\ 2\pi(j - x)/l & (x \leq j < x + l), \\ 2\pi & (j \geq x + l), \end{cases} \quad (73)$$

and define the local twist operator

$$\hat{U}_{x,l} = \exp \left[-i \sum_{j \in \mathbb{Z}} \theta_j (\hat{S}_j^{(z)} + S) \right] \quad (74)$$

which is supported on the interval $[x, x + l] \cap \mathbb{Z}$. The following result is due to Lieb, Schultz, and Mattis [LSM61]:

Lemma 4.7. *Let ω be a ground state of an $U(1)$ invariant Hamiltonian $\hat{H}$. Then there exist some constants C and l_0 , which depend on the Hamiltonian, that*

$$0 \leq \omega(\hat{U}_{x,l}^\dagger [\hat{H}, \hat{U}_{x,l}]) \leq \frac{C}{l} \quad (75)$$

for all $x \in \mathbb{R}$ and $l \geq l_0$.

Proof. See [Tas20]. □

The authors of [LSM61] were able to find low-lying states of the antiferromagnetic Heisenberg chain with half-integer spin. This proved that the system is gapless, in contrast with the integer spin case where a unique ground state is separated from the other states by the Haldane gap.

Lemma 4.7 equally applies to the systems with locally unique gapped ground states, but implies different consequences. If ω is such a ground state, then the twisted state $\hat{A} \mapsto \omega(\hat{U}_{x,l}^\dagger \hat{A} \hat{U}_{x,l})$ must converge to the same state with ω in the thermodynamic limit. This is the content of the following:

Lemma 4.8. *Let ω be a locally unique gapped ground state with energy gap $\Delta E > 0$ of a $U(1)$ invariant Hamiltonian $\hat{H}$. Then with the same constants C and l_0 as in Lemma 4.7, we have*

$$|\omega(\hat{U}_{x,l})|^2 \geq 1 - \frac{C}{l} \frac{1}{\Delta E} \quad (76)$$

for all $x \in \mathbb{R}$ and $l \geq l_0$.

Proof. Set $\hat{W} = \hat{U}_{x,l} - \omega(\hat{U}_{x,l})$ so that $\omega(\hat{W}) = 0$. Using the defining property of locally unique gapped ground state,

$$\omega(\hat{W}^\dagger [\hat{H}, \hat{W}]) \geq \Delta E \omega(\hat{W}^\dagger \hat{W}). \quad (77)$$

Substituting the definition of $\hat{W}$,

$$\omega(\hat{U}_{x,l}^\dagger [\hat{H}, \hat{U}_{x,l}]) \geq \omega(\hat{U}_{x,l}) \omega([\hat{H}, \hat{U}_{x,l}]) + \Delta E (1 - |\omega(\hat{U}_{x,l})|^2). \quad (78)$$

Now, the desired result would follow upon invoking the inequality (75), once we show that $\omega([\hat{H}, \hat{U}_{x,l}]) = 0$. Since every local operator is of the form $\hat{A} = (\hat{V}_1^\dagger \hat{V}_1 - a) + i(\hat{V}_2^\dagger \hat{V}_2 - b)$, it suffices to observe that

$$\omega([\hat{H}, \hat{V}^\dagger \hat{V}]) = \omega(\hat{H} \hat{V}^\dagger \hat{V}) - \omega(\hat{V}^\dagger \hat{V} \hat{H}) = \omega((\hat{V}^\dagger \hat{V} \hat{H})^\dagger) - \omega(\hat{V}^\dagger \hat{V} \hat{H}) = 0. \quad (79)$$

Note that $\omega(\hat{V}^\dagger \hat{V} \hat{H}) \in \mathbb{R}$, as implied by (69). $\square$

4.2 Generalized LSM theorem

Let ω be a locally unique gapped ground state with energy gap $\Delta E > 0$ of a $U(1)$ invariant Hamiltonian $\hat{H}$ on $\mathbb{Z}$. We denote by $\mathcal{T}_p$ the shift operator

$$\mathcal{T}_p(\hat{S}_j^{(\alpha)}) = \hat{S}_{j+p}^{(\alpha)}, \quad \alpha = x, y, z, \quad j \in \mathbb{Z}. \quad (80)$$

It is a linear $*$ -automorphism: $\mathcal{T}_p(\hat{A}\hat{B}) = \mathcal{T}_p(\hat{A})\mathcal{T}_p(\hat{B})$ and $\mathcal{T}_p(\hat{A}^\dagger) = \mathcal{T}_p(\hat{A})^\dagger$. Suppose that ω is invariant under translation by p in the sense that

$$\omega(\mathcal{T}_p(\hat{A})) = \omega(\hat{A}). \quad (81)$$

For any l larger than $l_1 = \max\{C/\Delta E, l_0\}$, Lemma 4.8 guarantees that $\omega(\hat{U}_{x,l}) \neq 0$ for all x . Since the assumption (81) implies that

$$\omega(\hat{U}_{0,l}) = \omega(\hat{U}_{p,l}), \quad (82)$$

the mapping $[0, p] \ni x \mapsto \omega(\hat{U}_{x,l})$ then defines a closed loop in $\mathbb{C} \setminus \{0\}$. We denote the winding number of this path by n_ω .

For a ground state satisfying (81), we define its filling factor as

$$\nu_\omega = \omega \left(\sum_{j=0}^{p-1} (\hat{S}_j^{(z)} + S) \right). \quad (83)$$

Note that the local operator $\hat{S}_j^{(z)} + S$ has integral eigenvalues, but ν_ω need not be an integer.

Lemma 4.9. *Let ω be a locally unique gapped ground state of a $U(1)$ invariant Hamiltonian $\hat{H}$, and assume that ω is invariant under translation by p . Then*

$$n_\omega = \nu_\omega. \quad (84)$$

Proof. Let $x_p \in \mathbb{R}$ be the smallest nonnegative number which is congruent to x modulo p . Note that $0 \leq x_p < p$ and $x_p - x \in p\mathbb{Z}$ by definition. Then

$$\hat{U}_{x,l} = \exp \left[-i \sum_{x \leq j < x+l} \frac{2\pi(j-x)}{l} (\hat{S}_j^{(z)} + S) \right] \quad (85a)$$

$$\begin{aligned} \mathcal{T}_{x_p-x}(\hat{U}_{x,l}) &= \exp \left[-i \sum_{x_p \leq j < x_p+l} \frac{2\pi(j-x)}{l} (\hat{S}_j^{(z)} + S) \right] \\ &= \exp \left[i \frac{2\pi x}{l} \sum_{x_p \leq j < x_p+l} (\hat{S}_j^{(z)} + S) \right] \exp \left[-i \sum_{x_p \leq j < x_p+l} \frac{2\pi j}{l} (\hat{S}_j^{(z)} + S) \right] \\ &= e^{i2\pi x \nu_{x_p,l}} \mathcal{T}_{x_p-x}(\hat{U}_{0,l}) \end{aligned} \quad (85b)$$

with

$$\hat{\nu}_{a,l} := \frac{1}{l} \sum_{a \leq j < a+l} (\hat{S}_j^{(z)} + S) \quad (0 \leq a < p). \quad (86)$$

Note that in the limit $l \rightarrow \infty$, we have $l = pN + O(1)$ for some large N , and

$$\omega(\hat{\nu}_{a,l}) = \frac{1}{p} \sum_{a \leq j < a+p} \omega(\hat{S}_j^{(z)} + S) + O(l^{-1}) = \frac{1}{p} \nu_\omega + O(l^{-1}), \quad (87)$$

where the sub-leading term can be bound by C/l with C independent of a because there are only finitely many values a take. Using this, we want to establish

$$\omega(\hat{U}_{x,l}) \approx e^{i2\pi x \nu_\omega / p} \omega(\hat{U}_{0,l}) \quad (88)$$

in the limit $l \rightarrow \infty$, which would show that the winding number of the curve $[0, p] \ni x \mapsto \omega(\hat{U}_{x,l})$ is ν_ω . To this end, we note that (85b) implies

$$\begin{aligned} & |\omega(\hat{U}_{x,l}) - e^{i2\pi x \nu_\omega / p} \omega(\hat{U}_{0,l})| \\ &= |\omega((e^{i2\pi x \hat{\nu}_{x,p,l}} - e^{i2\pi x \nu_\omega / p}) \hat{U}_{0,l})| \\ &\leq \omega((e^{i2\pi x \hat{\nu}_{x,p,l}} - e^{i2\pi x \nu_\omega / p})(e^{-i2\pi x \hat{\nu}_{x,p,l}} - e^{-i2\pi x \nu_\omega / p})) \omega(\hat{U}_{0,l}^\dagger \hat{U}_{0,l}) \end{aligned} \quad (89a)$$

$$\begin{aligned} &= 4\omega(\sin^2[2\pi x(\hat{\nu}_{x,p,l} - \nu_\omega/p)]) \\ &\leq 16\pi^2 x^2 \omega((\hat{\nu}_{x,p,l} - \nu_\omega/p)^2) \end{aligned} \quad (89b)$$

$$\leq 32\pi^2 x^2 \left\{ \omega([\hat{\nu}_{x,p,l} - \omega(\hat{\nu}_{x,p,l})]^2) + [\omega(\hat{\nu}_{x,p,l}) - \nu_\omega/p]^2 \right\}. \quad (89c)$$

In (89a) we applied Cauchy-Schwarz inequality to the hermitian form $(\hat{A}, \hat{B}) \mapsto \omega(\hat{A}^\dagger, \hat{B})$. (89b) is obtained by expanding $\sin^2(x)$ into an alternating series. (89c) is essentially the inequality $(A+B)^2 \leq 2(A^2+B^2)$. As will be stated in the next lemma, the first summand in the bottom line is small due to the *cluster property* of ω . The second summand was already shown to be small in (87). $\square$

Lemma 4.10. *Every pure state ω has the cluster property,*

$$\lim_{|x-y| \rightarrow \infty} [\omega(\hat{A}_x \hat{B}_y) - \omega(\hat{A}_x) \omega(\hat{B}_y)] = 0. \quad (90)$$

Proof. See [BR87] or [Str21]. $\square$

In particular,

$$\omega([\hat{\nu}_{a,l} - \omega(\hat{\nu}_{a,l})]^2) = \frac{1}{l^2} \sum_{j,k=\lceil a \rceil}^{\lceil a \rceil + l - 1} \omega([\hat{S}_j^{(z)} - \omega(\hat{S}_j^{(z)})][\hat{S}_k^{(z)} - \omega(\hat{S}_k^{(z)})]) \quad (91)$$

can be divided into two summations, $\sum_{|j-k| \geq D}$ and $\sum_{|j-k| < D}$, where D is such that $\omega(\cdots) < \epsilon$ whenever $|j-k| \geq D$. Then

$$\omega([\hat{\nu}_{a,l} - \omega(\hat{\nu}_{a,l})]^2) < \frac{1}{l^2} (l^2 \epsilon + O(l) D S^2) < 2\epsilon \quad (92)$$

for sufficiently large l . Since ϵ is arbitrary, the proof of Lemma 4.9 is now completed. We have thus established the following.

Theorem 4.11 (generalized LSM theorem). *Suppose $\hat{H}$ is a $U(1)$ invariant Hamiltonian and it has a locally unique gapped ground state ω . Further assume that ω is invariant under translation by p . Then the filling factor ν_ω must be an integer.*

Corollary 4.12 (Affleck-Lieb). *Let S be a half-odd integer. Then the antiferromagnetic Heisenberg chain does not have a unique gapped ground state.*

Proof. Global uniqueness implies local uniqueness and invariance under translation by 1. It remains to check that the filling factor is non-integer. Let us give an intuitive argument. As explored in Sec. 2.2, a staggered spin configuration gives a local minima of the action. Quantum mechanically, this corresponds to two different states $|\psi_1\rangle, |\psi_2\rangle$ related by

$$|\psi_2\rangle = \bigotimes_{n \in \mathbb{Z}} e^{i\pi \hat{S}_n^{(x)}} |\psi_1\rangle. \quad (93)$$

We can guess that the symmetric superposition $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ will have slightly lower energy than the antisymmetric superposition $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$, due to quantum tunneling as in symmetric double well. $|\psi_+\rangle$ will be our ground state, which is obviously invariant under translation by 1. Its filling factor is

$$\nu_\omega = \omega \left(\hat{S}_0^{(z)} + S \right) = S \in \mathbb{Z} + \frac{1}{2}. \quad (94)$$

□

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